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THE GALLAI–YOUNGER CONJECTURE FOR PLANAR GRAPHS

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Younger conjectured that for every k there is a $g(k)$ such that any digraph G without k vertex disjoint cycles contains a set X of at most $g(k)$ vertices such that $G-X$ has no directed cycles. Gallai had previously conjectured this result for $k=1$. We prove this conjecture for planar digraphs. Specifically, we show that if G is a planar digraph without k vertex disjoint directed cycles, then G contains a set of at most $O(k\log(k)\log(\log(k)))$ vertices whose removal leaves an acyclic digraph. The work also suggests a conjecture concerning an extension of Vizing's Theorem for planar graphs*.

1. Introduction and overview

A *cycle packing* in a digraph G is a set of vertex disjoint directed cycles of G. A *cycle cover* of a digraph G is a set of vertices X such that $G-X$ contains no directed cycles, i.e. $G-X$ is *acyclic*. The *cycle packing number* of G, denoted $cp(G)$, is the maximum cardinality of a cycle packing in G. The *cycle cover number* of G, denoted $cc(G)$, is the minimum cardinality of a cycle cover of G. Obviously, $cp(G)\leq cc(G)$. Gallai [1] conjectured that there is a constant K such that $cc(G)\leq K$ for any graph G with $cp(G)=1$. The analogous question for undirected graphs was resolved by Bollobás [2]. Younger [9] conjectured that there is some function g such that $cc(G)\leq g(cp(G))$. We refer to this as the Gallai–Younger Conjecture and note that it is the directed analogue of the earlier Erdős–Pósa Theorem. Note that the Gallai–Younger Conjecture is equivalent to the statement that there is a function h, going to infinity with n, such that every digraph G contains $h(cc(G))$ vertex disjoint directed cycles.

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* Since the time of submission the Gallai-Younger conjecture has been resolved for general graphs by Reed, Robertson, Seymour and Thomas. The bound given for general graphs is worse than exponential. Recently, Goemans and Williamson (IPCO, Vancouver, B.C., June 1996) have given a constant error factor between the parameters $cc(G), fcc(G)$ in the case of planar directed graphs G. Together with the results in this paper, this shows that $g(k)$ can be chosen as Ck for some constant C in the planar case.

Throughout the paper, all cycles are directed.

McCuaig [4] proved that a digraph with no two vertex disjoint cycles contains a cycle cover consisting of at most three vertices, thus settling the conjecture of Gallai. Hence the Gallai–Younger conjecture holds for graphs with $cp(G)=1$.

Thomassen [7] studied digraphs with large minimal outdegree (the outdegree of a vertex in a simple digraph is the number of arcs leaving it, the minimal outdegree of G, denoted $\delta^+(G)$, is the minimum of the outdegrees of its vertices). For any acyclic G we have $\delta^+(G)=0$, thus $cc(G)\geq\delta^+(G)$. Thomassen proved that there is a function f such that any simple digraph G with $\delta^+(G)\geq f(k)$ contains at least k vertex disjoint cycles (for a related conjecture, see [3]).

Metzlar and Murty [5] proved the Gallai–Younger conjecture for planar digraphs with maximum indegree and outdegree 2. Specifically, they showed that for any such graph $cc(G)<4cp(G)$.

Seymour [6] pointed out a relationship between the cycle cover number and the related fractional cycle cover number. A *fractional α cycle cover* is a weight function w from V to $[0,1]$ such that:

$$\sum_{v\in V} w(v) = \alpha$$

and each cycle C satisfies:

$$\sum_{v\in C} w(v) \geq 1.$$

A cycle cover is simply a fractional cycle cover in which each weight is 0 or 1. The *fractional cycle cover number* of G, denoted $fcc(G)$, is the smallest α for which G has a fractional α cycle cover. Clearly, $fcc(G)\leq cc(G)$. Note also that $fcc(G)$ is the optimal value for the linear program (LP)

$$\min\{1^T w : Aw \geq \mathbf{1}, w \geq 0\} \tag{1}$$

where A is the cycle-vertex incidence matrix for G. Seymour proved that

$$cc(G) = O(fcc(G)\log(fcc(G))\log(\log(fcc(G)))).$$

We are going to apply Seymour's result to prove the Gallai–Younger conjecture for planar digraphs. To do so, we need to introduce fractional cycle packings. A *fractional α cycle packing* in G is a weight function w from the set of cycles of G to [0,1] such that:

$$\sum_{C \text{ a cycle of } G} w(C) = \alpha,$$

and for every vertex v in G

$$\sum_{C\ni v} w(C) \leq 1.$$

A cycle packing is simply a fractional cycle packing in which every weight is 0 or 1. The fractional cycle packing number of G, denoted $fcp(G)$ is simply the maximum α for which G has a fractional α cycle packing. Obviously, $fcp(G) \geq cp(G)$. Note that $fcp(G)$ is the optimal value for the dual of the LP (1) and hence by LP duality, $fcp(G) = fcc(G)$. We shall prove:

Theorem. *Every planar digraph G satisfies $fcp(G) \leq 28cp(G)$.*

Applying LP duality and Seymour's result, we obtain:

Corollary. *Every planar digraph G satisfies*

$$cc(G) = O(cp(G)\log(cp(G))\log(\log(cp(G)))).$$

We remark that this theorem trivially implies the existence of a polynomial time dynamic programming algorithm to determine if an input planar digraph contains k vertex disjoint directed cycles for fixed k.

To prove the theorem, we actually study a special class of fractional cycle packings. To begin we note that for any digraph G, $fcp(G)$ is rational and there is a fractional $fcp(G)$ cycle packing in G using only rational weights (this is because we are solving an LP with integer coefficients). Now, by taking k to be the LCM of the non-zero denominators, we see that every weight is an integral multiple of $\frac{1}{k}$. Thus, we can express this cycle packing as an integer k and a multi-set of cycles $\mathscr{C} = \{C_1, C_2, \ldots, C_{k*fcp(G)}\}$. The weight of a cycle C is then simply (# of appearances of C in $\mathscr{C})/k$. We shall restrict our attention to cycle packings of this form from now on. We can and will also insist that the fractional $fcp(G)$ cycle packing contains only chordless cycles by "shortcutting" cycles with chords.

The other restriction we place on our fractional cycle packing relates to the planarity of G. We fix an embedding D of G in the plane and let F be the infinite face of D. Now, we can associate to each simple cycle of G, an inside and an outside in the standard way. We choose our cycle packing $(k, \mathscr{C} = \{C_1, \ldots, C_{k*fcp(G)}\})$ so that no two cycles in the packing cross, i.e., for any two cycles C_i and C_j in $\mathscr{C}$, either C_i does not intersect the inside of C_j or C_i does not intersect the outside of C_j. We call such a cycle packing *laminar*. We use the following lemma which we prove using standard uncrossing techniques.

Lemma 1. *Every planar graph G has a laminar fractional $fcp(G)$ cycle packing $(k, \mathscr{C} = \{C_1, \ldots, C_{k*fcp(G)}\})$*

From now on, we consider a planar graph G with an associated embedding D and a laminar fractional $fcp(G)$ cycle packing $(k, \mathscr{C} = \{C_1, \ldots, C_{k*fcp(G})\})$. We set $n = k*fcp(G)$. For two elements C_i and C_j in $\mathscr{C}$, we say that $C_i \leq C_j$ if C_i does not intersect the outside of C_j. This does not quite define a partial order because C_i and C_j may be copies of the same cycle. However, we can extend $\leq$ so that it is a partial order by arbitrarily ordering the copies of each cycle. We assume that we have done this. We now define the *depth* of an element C of $\mathscr{C}$ to be the length of the longest chain in this partial order which has C as its minimal element. Thus the depth of C is one more than the number of cycles in $\mathscr{C}$ which separate C from F.

A simple but important observation is:

Lemma 2. *If the depths of C_i and C_j differ by at least k, then these cycles are vertex disjoint.*

Now, for i at least 1, we define $\mathscr{C}_i$ to be the subset of elements of $\mathscr{C}$ with depths between $(i-1)*k+1$ and $i*k$ (inclusive). We let G_i be the (simple) graph obtained by taking the union of the elements of $\mathscr{C}_i$. Obviously, G_1,G_3, ... are disjoint. Similarly, $G_2,G_4,\ldots$ are disjoint. Further if G_i is non-empty, then each G_j with $j<i$ is also non-empty. Setting $r=cp(G)$, we see that this implies that G_j is empty for $j>2r$. For $1\leq i\leq 2r$, we let r_i be $cp(G_i)$.

The crux of the proof is the following result:

Lemma 3. *For each i, the total number of elements of $\mathscr{C}_i$ is at most $14r_ik$.*

Corollary. *Every planar graph G satisfies $fcp(G)\leq 28cp(G)$.*

Proof of Corollary. Since G_i and G_j are disjoint if i and j differ by two or more, we see that

$$\sum_{i=1}^{r} r_{2i-1} \leq r \quad \text{and} \quad \sum_{i=1}^{r} r_{2i} \leq r.$$

Thus, $\sum_{i=1}^{2r} r_i \leq 2r$. So,

$$|\mathscr{C}| = \sum_{i=1}^{2r} |\mathscr{C}_i| \leq \sum_{i=1}^{2r} 14r_ik \leq 28rk.$$

Since $|\mathscr{C}|=k*fcp(G)$, the result follows. ∎

This corollary is in fact the theorem we want to prove.

Instead of proving Lemma 3 we prove the following stronger lemma; note that it implies Lemma 3, since each family $\mathscr{C}_i$ defines a collection of contours as given in Lemma 4. In the statement of the lemma and elsewhere, a *nested family* of cycles is a laminar family of cycles which have pairwise intersecting insides. Equivalently they correspond to a chain in the partial order we defined on the cycles of $\mathscr{C}$. Note that this implies that for any two cycles C,C', either the inside of C is included in the inside of C', or conversely.

Lemma 4. *Let $\mathscr{C}$ be a non-empty laminar family of n simple contours $\{C_1,\ldots,C_n\}$ in the plane such that $\mathscr{C}$ contains no nested family of $k+1$ cycles and each point* *of the plane is in*[1] *at most k cycles of $\mathscr{C}$. Then, if $\mathscr{C}$ does not contain $r+1$ $(r \geq 1)$ pairwise disjoint contours, n is at most $(14r-11)k$.*

We may arbitrarily orient the contours of $\mathscr{C}$ to form a family of (directed) cycles and identify $\mathscr{C}$ with the planar (simple) graph G formed from the union of these cycles. We prove Lemma 4 by analyzing the structure of a vertex-minimal counterexample G. We first prove the following claims in the order given (the proofs are given in the following section).

Claim 1. *A minimal counterexample to* Lemma 4 *is connected, and hence strongly connected.*

Claim 2. *For any vertex v in a minimal counterexample G to* Lemma 4, *we have that $G-v$ is strongly connected.*

Claim 3. *Each cycle of $\mathscr{C}$ has length at least* 14.

If we remove three or fewer vertices from a minimal counterexample to Lemma 4 we obtain a connected digraph (though not necessarily a strongly connected one).

We also obtain:

Claim 4. *If we remove four vertices from a minimal counterexample to* Lemma 4 *we obtain a connected digraph.*

We now consider a vertex minimal counterexample $G, k, \mathscr{C}$ to Lemma 4 and a corresponding embedding D. We choose a maximum set $\mathscr{S}$ of vertex disjoint cycles of $\mathscr{C}$ with disjoint insides. Over all maximum size such sets, we choose one which minimizes the union of the insides of the elements of $\mathscr{S}$. Since $\mathscr{C}$ is a laminar family of cycles, we know that each element of $\mathscr{S}$ bounds a finite face of D.

We let $\mathscr{T}$ be the set of cycles of $\mathscr{C}$ which bound finite faces of D.

Our study of $\mathscr{T}$ is motivated by the following easily proved fact:

Claim 5. *Every element C of $\mathscr{C}-\mathscr{T}$ surrounds some element of $\mathscr{T}$, i.e., there is a $T \in \mathscr{T}$ in the union of C and the inside of C.*

We note that Claim 4 implies

Claim 6. *Every intersecting pair of elements of $\mathscr{T}$ intersects in a vertex or an edge.*

and

Claim 7. *If T_1, T_2, and T_3 are three pairwise-intersecting elements of $\mathscr{T}$, then either their common intersection is non-empty or there is a triangular face Φ of G such that for $i \neq j$, $T_i \cap T_j$ is a vertex of Φ.*

Now, we make an important observation whose proof is similar to that of Lemma 2.

[1] We identify a contour with its image.

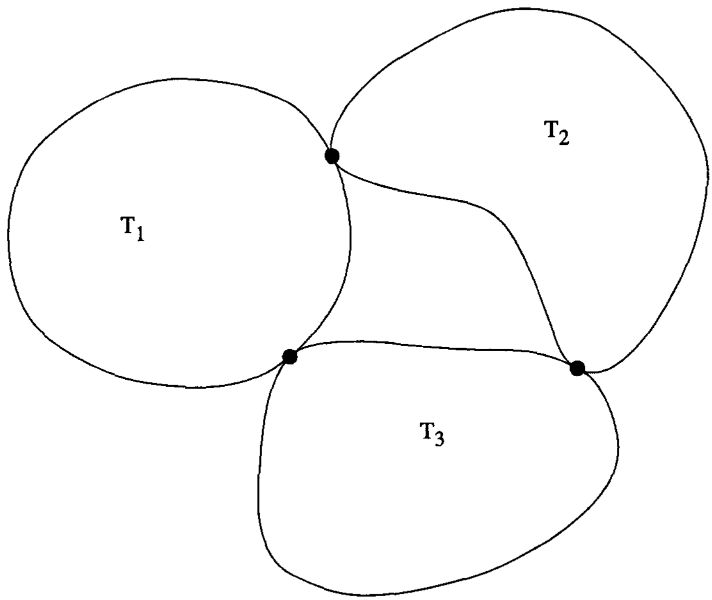

Fig. 1

Claim 8. *For every element S and T of $\mathcal{T}$, if C is an element of $\mathcal{C}$ which surrounds T but not S then C contains every element of $S \cap T$.*

Our assumptions on G also imply:

Claim 9. *For any vertex v of G, the set of cycles of $\mathcal{C}$ with v either on C or inside C has at most $2k$ elements.*

We now analyze the intersection pattern of the elements of $\mathcal{T}$ in more detail. To begin, we may use Claim 4 to prove:

Claim 10. *For any two elements S_1 and S_2 of $\mathcal{S}$ there are at most three elements of $\mathcal{T}$ which intersect both S_1 and S_2.*

Claim 10 and Euler's formula imply:

Claim 11. *The set $\mathcal{T}_2 = \{T | T \in \mathcal{T}, T$ intersects exactly two elements of $\mathcal{S}\}$ has at most $9r-9$ elements.*

Euler's formula also yields:

Claim 12. *The subset $\mathcal{T}_3$ of $\mathcal{T}$ consisting of those cycles which intersect at least three elements of S contains at most $2r-2$ elements.*

Finally, the maximality of $\mathcal{S}$ and Claims 6 and 7 imply:

Claim 13. *For each S in $\mathscr{S}$ there is a vertex such that every element of $\mathscr{I}-\mathscr{I}_2-\mathscr{I}_3$ intersecting S contains this vertex.*

Now, Claims 9, 11, 12, and 13 yield:

Claim 14. $|\mathscr{C}| \leq (14r-11)k$.

This completes the proof of Lemma 4. ∎

It remains only to give the proofs of Lemmas 1 and 2 and Claims 1–14. We do this in the next section. We remark that no effort was made to optimize the constant in this result. In fact, it is easy to see that combining the proofs of Claims 11 and 12 would yield a better constant. Lemma 4 suggests the following generalization of Vizing's Theorem for planar graphs.

Let $\mathscr{C}$ be a family of closed, self-avoiding contours[2] in the plane whose interiors are disjoint and whose pairwise intersections are either empty or a single point. Let ω be the maximum number of pairwise intersecting contours from this collection. Is there some constant C such that the contours can be coloured with at most $\omega+C$ colours so that no pair of intersecting contours have the same colour? In particular, we do not yet know of a counterexample to the case $C=1$. (A. Schrijver remarked that this problem is related to: given an Eulerian planar graph with a face two-colouring where the size of a largest black face is $k \geq 4$, $k+1$-colour the vertices so that the boundary of each 'black' face contains vertices of distinct colours.) To see how this relates to edge-colouring, one can translate this to the following problem by appropriately deforming the contours so that they have empty interiors. Let G be a planar graph and $\mathscr{P}$ a collection of noncrossing paths in G such that any pair $P,Q \in \mathscr{P}$ which intersect, intersect in a single vertex. Can the paths in $\mathscr{P}$ be $(\omega+C)$-coloured so that no intersecting pair of paths receive the same colour? Note that if $\mathscr{P}$ consists of the set of single edge paths and $C=1$, then this is equivalent to finding a $(\Delta+1)$-edge-colouring of G.

We remark on another special case. It was shown (see [8]) that any planar graph G is the intersection graph of a family of non-crossing circles in the plane, i.e., the vertices of G correspond to circles, and two vertices are adjacent if and only if their corresponding circles meet at a point. Thus in this special case, the contours can be 4-coloured.

2. The proofs

Proof of Lemma 1. We assume we are given an embedding D of a planar graph G and a fractional $fcp(G)$ cycle packing of the form: $(k, \mathscr{C}=\{C_1,\ldots,C_{k*fcp(G)}\})$. We set $n=k*fcp(G)$. We assume further that we have chosen such a fractional cycle packing with $\sum_{i=1}^{n}|E(C_i)|$ minimized. Here, $E(C_i)$ denotes the edge set of C_i. In

[2] We identify these contours by their range, rather than the actual real-valued function.

particular, this implies that all the cycles in $\mathscr{C}$ are simple. Subject to this, we choose $\mathscr{C}$ so as to minimize the total number of crossings[3] of the cycles within it (i.e. we minimize the sum over all (i,j) with $1 \le i < j \le n$, the number of crossings of C_i and C_j). We now show that such a fractional cycle packing is laminar. Assume not, then there are two cycles C_i and C_j which cross. Let H be the (plane embedded) multigraph obtained from the union of C_i and C_j by replicating any edge which appears in both C_i and C_j. Our choice of $\mathscr{C}$ ensures that any two edge disjoint cycles in H partition the edge set of H (in particular, configurations such as that depicted in Figure 2 cannot occur). It follows that for some l, we can partition $E(C_i) - E(C_j)$ into paths $P_1, \ldots, P_l$ and partition $E(C_j) - E(C_i)$ into paths $Q_1, \ldots, Q_l$ so that:

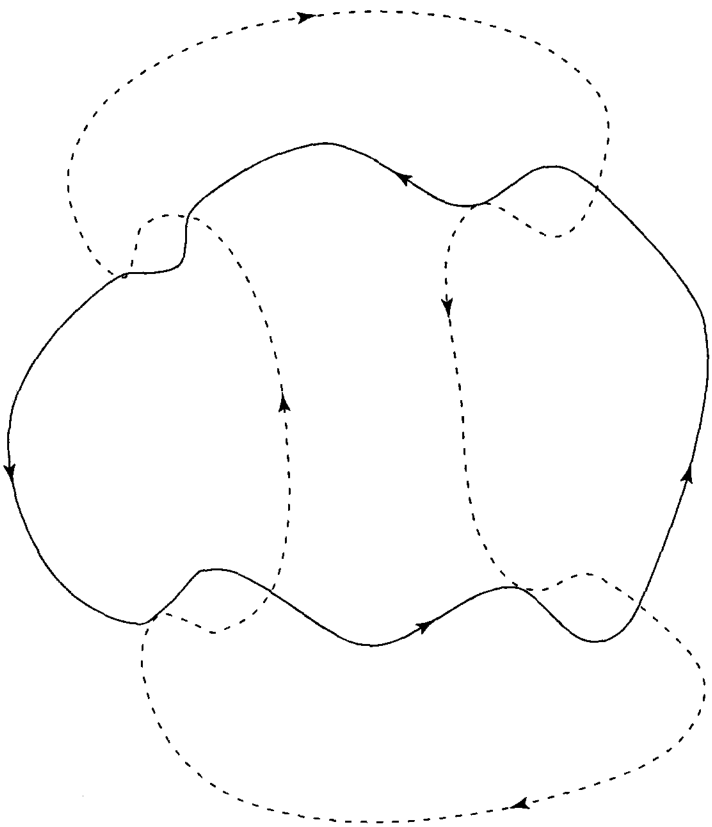

Fig. 2

[3] The number of crossings of C_i and C_j is the number of vertices v in $C_i \cup C_j$ for which the arcs of C_i and C_j incident to v, alternate.

(i) for each r, Q_r and P_r have the same endpoints, and

(ii) for any choice of r and s, P_r and Q_s are vertex disjoint except possibly for shared endpoints.

See Figure 3 for a pictorial representation of the situation.

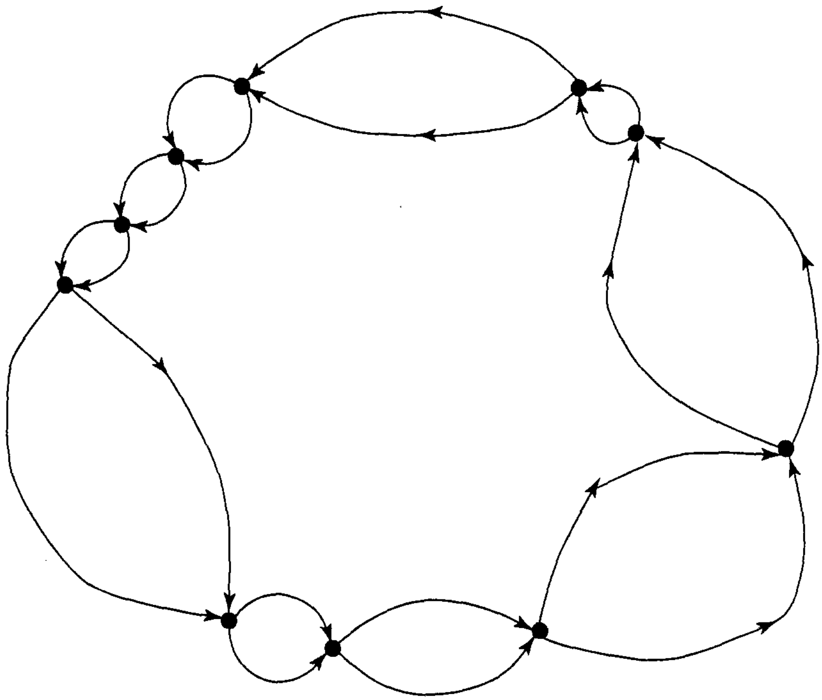

Fig. 3

Now, we can let C_i' be the cycle bounding the inner face in our drawing of H and let C_j' be the cycle bounding the outer face of our drawing of H. Obviously these two cycles do not cross. Furthermore, for any $m \notin \{i,j\}$ the sum of the number of crossings of C_m with C_i' and C_j' is the same as the sum of the number of crossings of C_m with C_i and C_j. But then, $\mathscr{C}-\{C_i,C_j\}+\{C_i',C_j'\}$ would contradict our assumption that $\mathscr{C}$ had as few crossings as possible. Thus, $\mathscr{C}$ is indeed laminar. ∎

Proof of Lemma 2 We can relabel the cycles so that C_i has depth d and C_j has depth at least $d+k$. We let S_1 be the set of elements of $\mathscr{C}$ which surround C_i (i.e. C is in S_1 if $C_i \leq C$) and S_2 be the set of elements of $\mathscr{C}$ which surround C_j. We let $S= S_2-S_1$. Since $|S_1|=d-1$ and $|S_2|\geq d+k-1$ we know $|S|\geq k$. We assume for a contradiction that there is a vertex v on $C_i \cap C_j$. We shall show that v is on every element of S. So, let C be any element of S. Obviously, v is not outside C because

C surrounds C_j. Similarly, v is not inside C because C_i surrounds C. So, v is on C. Thus v is on every element of S and also on C_j. But v can be on at most k cycles, so we have obtained the desired contradiction. ∎

Proof of Claim 1. Let $S_1,\ldots,S_l$ be the (strong) components of G. By the construction of G, each S_i contains a cycle. We let $\mathscr{C}_i = \{C \in \mathscr{C} : C \text{ is in } \cap S_i\}$ and let $r_i = cp(S_i)$. Obviously, $\mathscr{C} = \cup_{i=1}^{l} \mathscr{C}_i$ and $\sum_{i=1}^{l} r_i \leq r$. So, if $l \geq 2$ we can apply the induction hypothesis to each S_i to obtain $|\mathscr{C}_i| \leq (14r_i - 11)k$ and hence $|\mathscr{C}| \leq (14r-22)k$. Thus G could not have been a minimal counterexample. So, G is strongly connected. ∎

Proof of Claim 2. To see that $G-v$ does not contain two strong components which contain cycles we mimic the proof in Claim 1. The only difference is we have to take account of the cycles through v. However, there are only k of these so the bound of $(14r-22)k$ becomes $(14r-21)k$ which still contradicts our choice of G.

If $G-v$ has a trivial strong component, then there is a vertex u in G which is the tail of only one arc and this arc has head v. In this case, we contract the edge uv to obtain a new digraph G'. Also we obtain a new set of cycles $\mathscr{C}'$ by contracting uv in each cycle of $\mathscr{C}$ in which it appears. It is easy to verify that G', $\mathscr{C}'$ contradicts the minimality of G. ∎

Proof of Claim 3. If some cycle C of G has length less than 14, then removing C and all cycles intersecting C would produce a smaller counterexample. ∎

Proof of Claim 4. Let X be a set of vertices of G for which $G-X$ is disconnected.

Claim 4.1. $|X| > 3$

Proof of Claim 4.1. Suppose $|X| \leq 3$. As in the proof of Claim 2, we can show that at most one component of $G-X$ contains a cycle, as at most $3k$ elements of $\mathscr{C}$ pass through X. Further, at least one component of $G-X$ contains a cycle as otherwise every cycle in $\mathscr{C}$ passes through X so $|\mathscr{C}| \leq 3k$, a contradiction. Assume now that some component U of $G-X$ is acyclic. Then let P be a maximum length path in U and let s and t be the first and last vertices of P. By Claim 2, there are arcs from at least two vertices of X to s. Similarly, there are arcs to at least two vertices of X from t. Thus, there is a cycle in $X+U$. Now, there are at most $3k$ cycles of $\mathscr{C}$ which touch $X+U$ because any such cycle touches X. By induction, there are at most $(14(r-1)-11)k$ cycles of $\mathscr{C}$ which fail to hit $X+U$ (since $X+U$ contains a cycle). Thus $|\mathscr{C}| \leq (14r-22)k$, a contradiction. It follows that $G-X$ has only one component. ∎

So let X be a set of four vertices of G. As in the proof of Claim 4.1, we can show that exactly one component of $G-X$ contains a cycle. Furthermore, by Claim 4.1 and planarity, $G-X$ contains at most two components. Assume now that G contains an acyclic component U and define P, s, and t as before. We can show that $U+X$ is acyclic by mimicing the appropriate part of the proof of Claim 4.1.

Now, there are at least two vertices a_1 and a_2 in X which are tails of arcs with head s. Similarly, there are two vertices b_1 and b_2 of X which are heads of arcs with tail t. Since $X+U$ is acyclic, $X=\{a_1,a_2,b_1,b_2\}$. Furthermore, since $X+U$ is acyclic and each vertex of U has an outneighbour in $X+U-b_2$, it follows that there are no arcs from b_1 to U. Similarly, there are no arcs from b_2 to U or from U to a_1 or a_2.

Thus every path in $X+U$ with its endpoints in X has initial point a_1 or a_2 and endpoint b_1 or b_2. This yields four possible path types. There are only three of these types which are found as subpaths of cycles in $\mathscr{C}$ because of the non-crossing property of $\mathscr{C}$. We relabel X so that there is no subpath of an element of C from a_1 to b_2. We obtain a new graph H from G by deleting U and adding the arcs (a_1,b_1) (a_2,b_2) and (a_2,b_1). H is obviously planar. Furthermore, any cycle of $\mathscr{C}$ corresponds to a cycle of H, we simply replace subpaths through U by the appropriate new arc. So we have a set $\mathscr{C}'$ of cycles of H corresponding to $\mathscr{C}$. It is easy to verify that $(H,\mathscr{C}')$ contradicts the minimality of $(G,\mathscr{C})$. This contradiction shows that the claim holds. ∎

Proof of Claim 5. The proof is clear. ∎

Proof of Claim 6. If there were elements T_1 and T_2 of $\mathscr{T}$ whose intersection contained two non-adjacent vertices x and y, then $G-x-y$ would be disconnected, contradicting Claim 4. ∎

Proof of Claim 7. If Claim 7 failed to hold, we could find a set X contradicting Claim 4 in $(T_1\cap T_2)\cup(T_1\cap T_3)\cup(T_2\cap T_3)$. ∎

Proof of Claim 8. The proof follows the part of the proof of Lemma 2 in which we show each element of S contains all of $C_i\cap C_j$. ∎

Proof of Claim 9. The set of cycles $\{C\in\mathscr{C}|v \text{ is inside of } C\}$ forms a nested family. ∎

Proof of Claim 10. Now, if there were four elements of $\mathscr{T}$ intersecting both S_1 and S_2, then planarity yields a natural cyclic ordering of these four elements as T_1,T_2,T_3,T_4. If we let X contain exactly one element of $S_i\cap T_j$ for each (i,j) with $i\in(1,2)$ and $j\in(2,4)$, then G–X contains at least two components, one intersecting T_1-X and another intersecting T_3-X. Using Claim 3, we find a contradiction to Claim 4. ∎

Proof of Claim 11. We can form a planar multigraph whose edges are in 1-1 correspondence with $\mathscr{T}_2$ and whose vertices are in 1-1 correspondence with $\mathscr{S}$. Claim 10 ensures that this multigraph has maximum edge multiplicity at most 3. Euler's formula then ensures that it has at most $9|\mathscr{S}|-9$ edges. The result follows.[4] ∎

Proof of Claim 12. There is a bipartite simple planar graph in which one colour class corresponds to the elements of $\mathscr{S}$ and the other corresponds to the elements

[4] Note that the bound would be $9r-18$ but for the case $r=1$.

of $\mathcal{T}_3$. Each vertex of $\mathcal{T}_3$ has degree at least three in this graph. The claim follows because a bipartite planar graph on l vertices has at most $2l-2$ edges. ∎

Proof of Claim 13. By the maximality of $\mathcal{S}$, no two disjoint elements of $\mathcal{T}-\mathcal{T}_2-\mathcal{T}_3$ intersect S. If all such cycles meet at a vertex v, then v is our desired vertex. Otherwise there exists T_1,T_2 which meet at a vertex v_S which lies in a triangular face induced by v_S,v_1,v_2 where $\{v_i\}=T_i\cap S$. One now uses Claims 6, 7 to show that T_1,T_2 are the only elements of $\mathcal{T}-\mathcal{T}_2-\mathcal{T}_3$ intersecting S; thus v_S is the desired vertex. ∎

Proof of Claim 14. Let $\mathcal{T}'=\mathcal{S}\cup\mathcal{T}_2\cup\mathcal{T}_3$. Let $X=\{v_S:S\in\mathcal{S}\}$, where v_S is as given in Claim 13. By Claim 5, every element of $\mathcal{C}$ either surrounds some element of $\mathcal{T}'$ or surrounds or contains some element of X. Thus, $|\mathcal{C}|\le k*|\mathcal{T}'|+2k|X|$ (we apply Claim 9). The claim now follows directly from Claims 11–13. ∎

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